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Note

On generalized Fibonacci numbers and k-distance K_p-matchings in graphs

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ABSTRACT

In this paper, we give a new interpretation of the generalized Fibonacci numbers and the generalized Lucas numbers also with respect to the total number of K_p-matchings in the corona of special graphs. In special cases, we obtain the graph interpretation of the generalized Pell numbers and generalized Pell–Lucas numbers.

1. Introduction

For a general concept, see [1,2]. Consider a simple, undirected graph G with vertex set $V(G)$ and edge set $E(G)$. Let P_n and C_n denote an n-vertex path and an n-vertex cycle, respectively. By K_p, $p \geq 1$ we denote the complete graph on p vertices. The corona $G_1 \circ G_2$ of two graphs G_1 and G_2 is the graph obtained from the disjoint union of G_1 and n disjoint copies of G_2 (where n is the order of G_1) by joining the i-th vertex of G_1 to every vertex in the i-th copy of G_2. The line graph $L(G)$ of a graph G is the graph having vertex set $E(G)$ such that two vertices in $L(G)$ are adjacent if and only if their corresponding edges in G are adjacent. By $d_G(x, y)$ we denote the distance between vertices x and y in G. Let $k \geq 2$ be an integer. A subset $S \subset V(G)$ is a k-independent set of G if for each two distinct vertices $x, y \in S$, $d_G(x, y) \geq k$. In addition, a subset containing only one vertex and the empty set also are k-independent. If $k = 2$, then this definition reduces to the definition of an independent set in the classical sense. Let $NI_k(G)$ denote the number of k-independent sets in G and for $k = 2$, $NI_2(G) = NI(G)$. The parameter $NI(G)$ appears in the mathematical literature in the paper of Prodinger and Tichy; see [8]. They showed that $NI(P_n) = F_n$, where F_n is the n-th Fibonacci number defined by $F_0 = 1$, $F_1 = 2$ and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$. Moreover, they proved that $NI(C_n) = L_n$, where L_n is the n-th Lucas number defined by $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$, for $n \geq 2$.

Independently Merrifield and Simmons introduced the parameter $NI(G)$ (which they called σ-index) to the chemical literature; see [7]. They showed the correlation between this index and some physicochemical properties of a molecular graph. This index is topological and it is used to give the quantitative properties of a molecular graph.

Another very important graph parameter is the Hosoya index (the Merrifield–Simmons index closely related to the Hosoya index) defined as the number of all matchings in graph G, including the empty matching. Since these indices were introduced by chemists (see [7]), many papers have been published with the various applications in chemistry, e.g., the correlation of the boiling points of alkanes. Furthermore, the Hosoya index is connected to the monomer–dimer model of statistical physics [5]. The literature includes many papers dealing with the theory of counting of independent sets in graphs, the last survey written by Gutman and Wagner [4] collects and classifies these results; most of them are obtained quite recently.

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Table 1
The values of $F(k, n)$ and $L(k, n)$ for special case of n and k.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
F_n	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987	1597
$F(3, n)$	1	2	3	4	6	9	13	19	28	41	60	88	129	189	277	406
$F(4, n)$	1	2	3	4	5	7	10	14	19	26	36	50	69	95	131	181
$F(5, n)$	1	2	3	4	5	6	8	11	15	20	26	34	45	60	80	106
L_n	2	1	3	4	7	11	18	29	47	76	123	199	322	521	843	1364
$L(3, n)$	1	2	3	4	5	6	10	15	21	31	46	67	98	144	211	309
$L(4, n)$	1	2	3	4	5	6	7	8	13	19	26	34	47	66	92	126

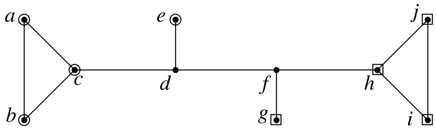

Fig. 1. Induced H-matching $\{\{a, b, c, e\}, \{i, j, g, h\}\}$ for $H = C_3 \cup K_1$.

Besides the usual Fibonacci and Lucas numbers many kinds of generalizations of these numbers have been presented in the literature. In [6], Kwaśnik and Włoch introduced the concept of the generalized Fibonacci numbers $F(k, n)$ and the generalized Lucas numbers $L(k, n)$ which give the number of all k-independent sets in graphs P_n and C_n, respectively.

The generalized Fibonacci numbers were defined by the following recurrence relation

$$\begin{aligned} &F(k, n) = n + 1 \quad \text{for } n = 0, 1, \dots, k \quad \text{and} \\ &F(k, n) = F(k, n-1) + F(k, n-k), \quad \text{for } n \geq k + 1. \end{aligned} \tag{1.1}$$

Moreover, the generalized Lucas numbers were given in the following way

$$\begin{aligned} &L(k, n) = n + 1, \quad \text{for } n = 0, 1, \dots, 2k-1 \quad \text{and} \\ &L(k, n) = (k-1)F(k, n-(2k-1)) + F(k, n-(k-1)), \quad \text{for } n \geq 2k. \end{aligned} \tag{1.2}$$

Recently, a more comfortable recurrence relation for the generalized Lucas numbers was given (see [10]), namely $L(k, n) = L(k, n-1) + L(k, n-k)$, for $n \geq 2k$.

It is easy to see that $F(2, n) = F_n$, for $n \geq 0$ and $L(2, n) = L_n$, for $n \geq 3$.

Table 1 gives the initial words of the generalized Fibonacci numbers and the generalized Lucas numbers for special case of n and k.

In the graph terminology for an arbitrary $k \geq 2$ we have

$$F(k, n) = NI_k(P_n) \quad \text{for } n \geq 1 \quad \text{and} \quad L(k, n) = NI_k(C_n) \quad \text{for } n \geq 3.$$

We can observe that for $n = 0, 1, 2$ and $k \geq 2$ the numbers $L(k, n)$ does not have the graph interpretation with respect to the number of k-independent sets of C_n. The generalized Fibonacci numbers and the generalized Lucas numbers are studied recently mainly with respect to the number of k-independent sets in graphs and their products (see [13]). In a special graph product, the number of k-independent sets is expressed using the concept of the generalized Fibonacci polynomial of a graph; see [9,11]. The concept of k-independent sets in graphs is studied in the literature in many papers, along with the concept of (k, l)-kernels in graphs; see [3,9].

In this paper, we give a new interpretation of the generalized Fibonacci numbers and the generalized Lucas numbers. First we apply this generalization to the counting of special families of subsets of the set of integers. Next we give the graph interpretation of these families, the generalized Fibonacci numbers and the generalized Lucas numbers. This interpretation is connected with the concept of k-distance H-matching of graphs. This concept generalizes the concept of H-matching of graphs.

One way to generalize both matchings and independent sets is the concept of H-matchings. Let G and H be two graphs; then an H-matching M of G is a subgraph of G such that all connected components of M are isomorphic to H. If M is also an induced subgraph of G, the H-matching is called induced (see Fig. 1 for example).

It is easy to see that if $H = K_2$, then K_2-matching is a matching. Moreover, if $H = K_1$, then an induced K_1-matching is an independent set; see a survey [14].

In this paper, we introduce the concept of k-distance H-matchings. Let $k \geq 1$. Let G and H be two graphs; then a k-distance H-matching M of G is a subgraph of G such that all connected components of M are isomorphic to H and for each two components H_1, H_2 from M and for each $x \in V(H_1)$ and $y \in V(H_2)$, $d_G(x, y) \geq k$. Note that if $k = 1$ and $H = K_2$, then this definition reduces to the definition of matching in the classical sense. If $k = 2$ and $H = K_1$, then we obtain the definition of independent sets in the classical sense.

2. Main results

Let $n \geq 1, p \geq 1$ be integers. Let $X = \{1, 2, \dots, \max\{p, n\}\}$ be the set of integers. Let $\mathcal{X}$ be a family of subsets of X such that $\mathcal{X} = \{\mathcal{Y}_i; i = 1, \dots, n\}$, where $\mathcal{Y}_i = \{\{i\}, \{i, r\}; r = 1, \dots, p-1\}$.

For every $i = 1, \dots, n$ the $\mathcal{Y}_i$ is named as the i-subfamily. From the definition of i-subfamily it follows that $\mathcal{Y}_i \cap \mathcal{Y}_j = \emptyset$ for every $1 \leq i, j \leq n, i \neq j$.

Let $k \geq 1$ be an integer. Let $\mathcal{Y} \subset \mathcal{X}$ be family of i-subfamilies such that

(i) $|\mathcal{Y}| = t$, for fixed $t \geq 0$ and
(ii) for each $\mathcal{Y}_i, \mathcal{Y}_j \in \mathcal{Y}$ holds $|i-j| \geq k$.

By $i(k, n, t)$, we denote the number of all subfamilies $\mathcal{Y}$ having exactly t, $t \geq 0$, i-subfamilies and further let $I(k, n) = \sum_{t \geq 0} i(k, n, t)$.

From this definition, it immediately follows that $I(1, n) = 2^n$. Hence in the further considerations, we assume that $k \geq 2$.

Theorem 2.1. *Let $k \geq 2, n \geq 1, t \geq 0$ be integers. Then $i(k, n, 0) = 1$, $i(k, n, 1) = n$. Let $t \geq 2$. If $n \leq k$, then $i(k, n, t) = 0$ and for $n \geq k+1$ we have the recurrence relation $i(k, n, t) = i(k, n-1, t) + i(k, n-k, t-1)$.*

Proof. The initial conditions are obvious. Assume now that $n \geq k+1$ and $t \geq 2$. Let $\mathcal{Y} \subset \mathcal{X}$. We recall that $\mathcal{Y}$ contains exactly t i-subfamilies of $\mathcal{X}$ such that for each $\mathcal{Y}_i, \mathcal{Y}_j \in \mathcal{Y}, i \neq j, |i-j| \geq k$.

Let $i_{\{\mathcal{Y}_n\}}(k, n, t)$ (respectively $i_{-\{\mathcal{Y}_n\}}(k, n, t)$) be the number of t-element subfamilies $\mathcal{Y}$ such that $\mathcal{Y}_n \in \mathcal{Y}$ (respectively $\mathcal{Y}_n \notin \mathcal{Y}$).

Then $i(k, n, t) = i_{\{\mathcal{Y}_n\}}(k, n, t) + i_{-\{\mathcal{Y}_n\}}(k, n, t)$ is the basic rule for counting of the families $\mathcal{Y}$. Consequently, we consider two cases.

1. $\mathcal{Y}_n \notin \mathcal{Y}$.

Then the definition of the family $\mathcal{Y}$ implies that we can find the families $\mathcal{Y}$ in the family $\mathcal{X} \setminus \mathcal{Y}_n$, which corresponds the family $\mathcal{X}'$ of i-subfamilies of the set $X' = \{1, \dots, \max\{p, n-1\}\}$, for an arbitrary $p \geq 1$ such that $\mathcal{X}' = \mathcal{X} \setminus \mathcal{Y}_n$. Then every t-elements, $t \geq 2$, family $\mathcal{Y}$ which satisfies the condition (ii) is t-elements family $\mathcal{Y}$ from $\mathcal{X}'$. Hence $i_{-\{\mathcal{Y}_n\}}(k, n, t) = i(k, n-1, t)$.

2. $\mathcal{Y}_n \in \mathcal{Y}$.

Then it is clear that $\mathcal{Y}_{n-i} \notin \mathcal{Y}$, for each $i = 1, \dots, k-1$. This implies that $\mathcal{Y} = \mathcal{Y}^* \cup \mathcal{Y}_n$ where $\mathcal{Y}^*$ is an arbitrary $(t-1)$ element family satisfying the condition (ii) of the family $\mathcal{X} \setminus \{\mathcal{Y}_{n-i}; i = 0, \dots, k-1\}$ which is isomorphic to the family $\mathcal{X}^*$ of subsets of the set $X = \{1, \dots, \max\{n-k, p\}\}$, such that $\mathcal{X}^* = \mathcal{X} \setminus \{\mathcal{Y}_{n-i}; i = 0, \dots, k-1\}$. Then $i_{\{\mathcal{Y}_n\}}(k, n, t) = i(k, n-k, t-1)$.

In consequence for the numbers $i(k, n, t)$, we obtain from the above cases the k-th order linear recurrence on the form $i(k, n, t) = i(k, n-1, t) + i(k, n-k, t-1)$.

Thus the theorem is proved. $\square$

Theorem 2.2. *Let $k \geq 2, n \geq 1$ be integers. Then $I(k, n) = F(k, n)$.*

Proof. If $n = 1, \dots, k$, then $t = 0$ or $t = 1$. Hence $I(k, n) = i(k, n, 0) + i(k, n, 1) = n + 1$ by Theorem 2.1 and $I(k, n) = F(k, n)$ for this case.

Let $n \geq k+1$. Then

$$\begin{aligned} I(k, n) &= \sum_{t \geq 0} i(k, n, t) = i(k, n, 0) + i(k, n, 1) + \sum_{t \geq 2} i(k, n, t) \\ &= 1 + n + \sum_{t \geq 2} (i(k, n-1, t) + i(k, n-k, t-1)) \\ &= 1 + n + \sum_{t \geq 2} i(k, n-1, t) + \sum_{t \geq 2} i(k, n-k, t-1) \\ &= 1 + n - (1 + n - 1) + \sum_{t \geq 0} i(k, n-1, t) - 1 + \sum_{t \geq 0} i(k, n-k, t) \\ &= \sum_{t \geq 0} i(k, n-1, t) + \sum_{t \geq 0} i(k, n-k, t) \\ &= I(k, n-1) + I(k, n-k). \end{aligned}$$

Consequently the number $I(k, n)$ satisfies the following recurrence relation

$$\begin{aligned} &I(k, n) = n + 1, \quad \text{for } n = 1, \dots, k \quad \text{and for } n \geq k+1 \\ &I(k, n) = I(k, n-1) + I(k, n-k). \end{aligned}$$

Then by (1.1) we have that $I(k, n) = F(k, n)$.

Thus the theorem is proved. $\square$

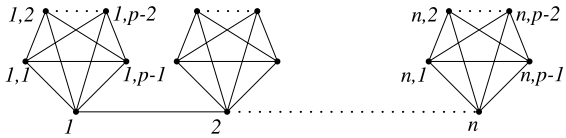

Fig. 2. Graph $G_n = P_n \circ K_{p-1}$.

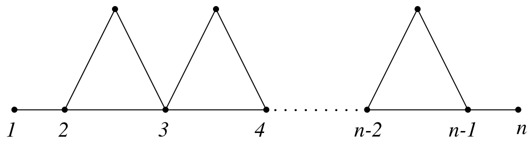

Fig. 3. Graph H_n.

Table 2
The values of $P(k, n)$ for special case of n and k.

n	3	4	5	6	7	8	9	10	11	12	13	14	15
P_n	5	12	29	70	169	408	985	2378	5741	13 860	33 461	80 782	195 025
$P(3, n)$	4	7	13	24	44	81	149	274	504	927	1 705	3 136	5 768
$P(4, n)$	4	6	9	15	25	40	64	104	169	273	441	714	1 156
$P(5, n)$	4	6	8	11	17	27	41	60	88	132	200	301	449

The family $\mathcal{X}$ can be represented as the vertex set of the graph G_n of order pn such that $G_n = P_n \circ K_{p-1}$, in Fig. 2, where vertices from $V(G_n)$ are labeled by integers belonging to a corresponding subset from $\mathcal{X}$.

Consequently in the graph terminology for an arbitrary $p \geq 3$, $k \geq 2$ the number $F(k, n)$, for $n \geq 1$ is equal to the total number of k-distance K_p-matchings of the graph $P_n \circ K_{p-1}$. If $p = 1$, then $P_n \circ K_{p-1}$ is isomorphic to P_n and we obtain the known interpretation from [6]. In particular if $k = 2$ and $p = 1$, then we obtain the classical result of Prodinger and Tichy from [8]. Note that if $p = 2$, then every K_2-matching is a matching and the number $F(k, n)$ does not have a graph interpretation with respect to the number of k-distance matchings in the graph $P_n \circ K_1$.

Now we consider this special case and we shall show that the number of k-distance matchings of $P_n \circ K_1$ is given by the generalized Pell numbers. The generalized Pell numbers $P(k, n)$ were introduced by Włoch; see [12]. They were defined by the following recurrence relation. Let $k \geq 2$, $n \geq 3$ be integers. Then $P(k, n) = 2n - 2$ for $n \leq k$,

$$P(k, k+1) = 2k + 1,$$
$$P(k, k+2) = \begin{cases} 12 & \text{if } k = 2 \\ 2k+7 & \text{if } k \geq 3, \end{cases} \quad \text{and for } n \geq k+3$$
$$P(k, n) = P(k, n-k+1) + P(k, n-1) + P(k, n-k).$$

Table 2 shows the initial words for the generalized Pell sequence for special n and k.

If $k = 2$, then $P(2, n)$ is the Pell number $\mathbf{P}_n$ given by known recurrence relation

$$\mathbf{P}_n = 2\mathbf{P}_{n-1} + \mathbf{P}_{n-2} \quad \text{for } n \geq 5 \quad \text{and} \quad \mathbf{P}_3 = 5, \qquad \mathbf{P}_4 = 12. \tag{1.3}$$

In the graph terminology the number $P(k, n)$ for $n \geq 3$, $k \geq 2$ is equal to the total number of k-independent sets of the graph H_n, given on Fig. 3, i.e., $NI_k(H_n) = P(k, n)$.

Considering the graph $P_n \circ K_1$ it is easy to observe that its line graph $L(P_n \circ K_1)$ is isomorphic to graph H_{n+1}. Moreover, every k-distance matching, $k \geq 1$ of $P_n \circ K_1$ corresponds to a $(k+1)$-independent set of the graph H_{n+1}.

This implies that for $p = 2$ and $k \geq 1$ holds $I(k, n) = P(k+1, n+1)$, for $n \geq 3$.

Now we consider another family of subsets of the set X.

Let $k \geq 1$ be an integer. Let $\mathcal{F} \subset \mathcal{X}$ be a family of i-subfamilies such that

(iii) $|\mathcal{F}| = t$, for fixed $t \geq 0$ and

(iv) for each $\mathcal{Y}_i, \mathcal{Y}_j \in \mathcal{F}$ holds $k \leq |i - j| \leq n - k$.

By $j(k, n, t)$, we denote the number of all families $\mathcal{F}$ having exactly t, $t \geq 0$, i-subfamilies and next let $J(k, n) = \sum_{t \geq 0} j(k, n, t)$.

From the definition, it follows that $J(1, n) = 2^n$. Hence in the next considerations let $k \geq 2$.

Theorem 2.3. *Let $k \geq 2$, $n \geq 1$ be integers. Then $j(k, n, 0) = 1$, $j(k, n, 1) = n$. Let $t \geq 2$. Then for $n < kt$, $j(k, n, t) = 0$. For $t \geq 2$ and $n \geq kt$ we have $j(k, n, t) = (k-1)i(k, n-(2k-1), t-1) + i(k, n-(k-1), t)$.*

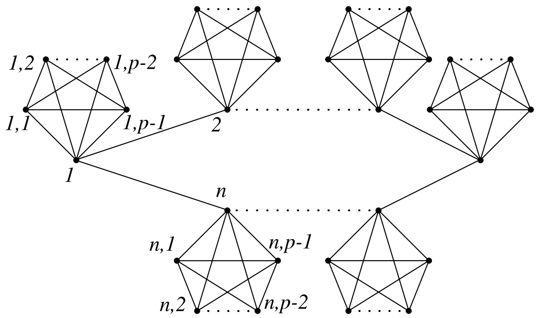

Fig. 4. Graph $R_n = C_n \circ K_{p-1}$.

Proof. The initial conditions are obvious. Assume that $n \geq kt$ and $t \geq 2$. Let $\mathcal{F} \subset \mathcal{X}$ contain exactly t i-subfamilies such that for each $\mathcal{Y}_i, \mathcal{Y}_j \in \mathcal{F}$ holds $k \leq |i-j| \leq n-k$. Without loss of generality consider subfamilies $\mathcal{Y}_1, \mathcal{Y}_2, \ldots \mathcal{Y}_{k-1}$. To calculate the number of $\mathcal{F}$, we consider the following possibilities.

1. $\mathcal{Y}_i \notin \mathcal{F}$ for $i = 1, \ldots, k-1$.

Then for each $l, j \in \{k, k+1, \ldots, n\}$, holds $|l-j| \leq n-k$. Thus we deduce from this fact that to construct the family $\mathcal{F}$ on t i-subfamilies we can choose these i-subfamilies from $n-(k-1)$ i-subfamilies of $\mathcal{X}$. If $\mathcal{Y}_l, \mathcal{Y}_j \in \mathcal{F}$, then the condition $|l-j| \geq k$ must be satisfied. Then by Theorem 2.1 we obtain that the number of all t-element families $\mathcal{F}$ not containing $\mathcal{Y}_i$, $i = 1, \ldots, k-1$ is equal to $i(k, n-(k-1), t)$.

2. $\mathcal{Y}_i \in \mathcal{F}$ for $1 \leq i \leq k-1$.

Suppose now that $\mathcal{Y}_j \in \mathcal{F}$ and $j \neq i$. Then it is clear that $j > i$ and the condition $k \leq |i-j| \leq n-k$ is equivalent to $i+k \leq j \leq n-k+i$. This gives that the remaining $t-1$ i-subfamilies $\mathcal{Y}_j$, $\mathcal{Y}_j \neq \mathcal{Y}_i$ we have to choose from $n-(2k-1)$ i-subfamilies of X. Theorem 2.1 gives that there is exactly $i(k, n-(2k-1), t-1)$ possibilities. Because the subfamily $\mathcal{Y}_i$ can be chosen on $k-1$ ways, so there is exactly $(k-1)i(k, n-(2k-1), t-1)$ subfamilies $\mathcal{F}$ in this case.

Finally from the above cases we obtain that $i(k, n, t) = (k-1)i(k, n-(2k-1), t-1) + i(k, n-(k-1), t)$, which completes the proof. $\square$

Using the same method as in Theorem 2.2 we can prove the following theorem.

Theorem 2.4. *Let $k \geq 2$, $n \geq 1$ be integers. Then $J(k, n) = L(k, n)$.*

Now the family $\mathcal{X}$ can be represented as the vertex set of the graph $R_n = C_n \circ K_{p-1}$, in Fig. 4, where vertices from $V(R_n)$ are labeled by integers belonging to corresponding subsets from $\mathcal{X}$.

In the graph interpretation for an arbitrary $p \geq 3$, $k \geq 2$ the number $L(k, n)$, $n \geq 3$, is equal to the number of all k-distance K_p-matchings of the graph $C_n \circ K_{p-1}$.

If $p = 1$, then the graph $R_n = C_n \circ K_{p-1}$ is isomorphic to C_n and the known interpretation from [6] immediately follows. In addition if $k = 2$, then we obtain the classical result from [8].

Analogously as for the graph G_n, if $p = 2$ then every K_2-matching is a matching and the number $L(k, n)$ does not have a graph interpretation with respect to the total number of k-distance matchings of the graph $C_n \circ K_{p-1}$. In this special case the number of k-distance K_2-matchings of $C_n \circ K_2$ is given by the generalized Pell–Lucas numbers.

The generalized Pell–Lucas numbers $Q(k, n)$ were introduced in [12] together with generalized Pell numbers $P(k, n)$. They were defined by the following recurrence relation.

Let $k \geq 2$, $n \geq 3$ be integers. Then

$$Q(k, n) = 2n + 1 \quad \text{for } n \leq 2k-3$$

$$Q(k, 2k-2) = 5k-4,$$

$$Q(k, 2k-1) = \begin{cases} 14 & \text{if } k = 2 \\ 8k-3 & \text{if } k \geq 3, \end{cases} \quad \text{and for } n \geq 2k$$

$$Q(k, n) = k \cdot P(k, n-2k+3) + (k-1) \cdot P(k, n-2k+4) + P(k, n-k+2). \tag{1.4}$$

If $k = 2$ and $n \geq 3$, then $Q(2, n)$ is the Pell–Lucas number Q_n.

Table 3 shows the initial words for the generalized Pell–Lucas sequence for special n and k. In the graph interpretation (see [12]), the number $Q(k, n)$ for $k \geq 2$, $n \geq 3$ is equal to the number of all k-independent sets of the graph T_n, given on Fig. 5, i.e., $NI_k(T_n) = Q(k, n)$.

Considering the graph $C_n \circ K_1$ it is easy to observe that $L(C_n \circ K_1)$ is isomorphic to graph T_n. In addition every k-distance matching, $k \geq 1$ of a graph $C_n \circ K_1$ corresponds to $(k+1)$-independent set of the graph T_n. Consequently for $p = 1$ and $k \geq 1$, $J(k, n) = Q(k+1, n)$, for $n \geq 3$.

Note that in [12] the generalized Pell–Lucas numbers $Q(k, n)$ were given using the generalized Pell numbers $P(k, n)$. In this paper, we give a more comfortable identity for the numbers $Q(k, n)$, for $n \geq 2k$.

Table 3
The values of $Q(k, n)$ for special case of n and k.

n	3	4	5	6	7	8	9	10	11	12	13	14	15
Q_n	6	14	34	82	198	478	1154	2786	6726	16 238	39 202	94 642	2 28 486
$Q(3, n)$	7	11	21	39	71	131	241	443	815	1 499	2 757	5 071	9 327
$Q(4, n)$	7	9	11	16	29	49	76	121	199	324	521	841	1 364
$Q(5, n)$	7	9	11	13	15	21	37	61	89	125	183	281	431

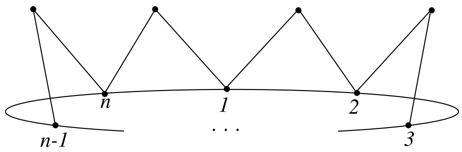

Fig. 5. Graph T_n.

Theorem 2.5. *Let $k \geq 2$, $n \geq 3$ be integers. Then*

$$Q(k, n) = 2n + 1 \quad \text{for } n \leq 2k - 3$$
$$Q(2k - 2, k) = 5k - 4,$$
$$Q(2k - 1, k) = \begin{cases} 14 & \text{if } k = 2 \\ 8k - 3 & \text{if } k \geq 3, \end{cases} \quad \text{and for } n \geq 2k$$
$$Q(k, n) = Q(k, n - 1) + Q(k, n - k) + Q(k, n - k + 1).$$

Proof. The initial conditions follow from (1.4). Let $n \geq 2k$. Then using recurrence formula (1.4) for the generalized Pell–Lucas numbers $Q(k, n)$ we have

$$\begin{aligned}
&Q(k, n - 1) + Q(k, n - k) + Q(k, n - k + 1) \\
&\quad = kP(k, n - 1 - 2k + 3) + (k - 1)P(k, n - 1 - 2k + 4) + P(k, n - 1 - k + 2) \\
&\qquad + kP(k, n - k - 2k + 3) + (k - 1)P(k, n - k - 2k + 4) + P(k, n - k - k + 2) \\
&\qquad + kP(k, n - k + 1 - 2k + 3) + (k - 1)P(k, n - k + 1 - 2k + 4) + P(k, n - k + 1 - k + 2) \\
&\quad = kP(k, n - 2k + 2) + (k - 1)P(k, n - 2k + 3) + P(k, n - k + 1) \\
&\qquad + kP(k, n - 3k + 3) + (k - 1)P(k, n - 3k + 4) + P(k, n - 2k + 2) \\
&\qquad + kP(k, n - 3k + 4) + (k - 1)P(k, n - 3k + 5) + P(k, n - 2k + 3) \\
&\quad = k[P(k, n - 2k + 2) + P(k, n - 3k + 3) + P(k, n - 3k + 4)] \\
&\qquad + (k - 1)[P(k, n - 2k + 3) + P(k, n - 3k + 4) + P(k, n - 3k + 5)] \\
&\qquad + P(k, n - k + 1) + P(k, n - 2k + 2) + P(k, n - 2k + 3) \\
&\quad = k[P(k, (n - 2k + 3) - k + 1) + P(k, (n - 2k + 3) - 1) + P(k, (n - 2k + 3) - k)] \\
&\qquad + (k - 1)[P(k, (n - 2k + 4) - 1) + P(k, (n - 2k + 4) - k) + P(k, (n - 2k + 4) - k + 1)] \\
&\qquad + P(k, (n - k + 2) - 1) + P(k, (n - k + 2) - k) + P(k, (n - k + 2) - k + 1) \\
&\quad = kP(k, n - 2k + 3) + (k - 1)P(k, n - 2k + 4) + P(k, n - k + 2) = Q(k, n).
\end{aligned}$$

Thus the theorem is proved. □

It is easy to see that for $n \geq 3$ and $k = 2$, we obtain known recurrence formula for the Pell–Lucas numbers, namely $Q_n = 2Q_{n-1} + Q_{n-2}$ with initial conditions $Q_3 = 6$, $Q_4 = 14$.

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